OCR
Oxford Cambridge and RSA

Monday 05 October 2020

AS Level Further Mathematics B (MEI)

Y410/01 Core Pure

Worked Solutions

Printed Answer Booklet

Time allowed: 1 hour 15 minutes

You must have:
- Question Paper Y410/01 (inside this document)
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

R red level
- longer questions (6+ marks)
- higher level problem solving
- hard AS content

A amber level
- Shorter questions (3-6 marks)
- low level problem solving
- harder AS content.

G green level
- Short questions (1-3 marks)
- minimal problem solving
- easier AS content.

E explanation

Q1: Series
Q2: Complex numbers
Q3: Complex numbers, algebra
Q4: Matrices, transformations
Q5: Proof
Q6: Matrices
Q7: Algebra, complex numbers
Q8: Matrices, transformations
Q9: Matrices, vectors
Q10: Vectors

Grade Boundaries

Grade	A	B	C	D	E	U
Mark / 144	37	31	25	20	15	0

Turn over

1 **In this question you must show detailed reasoning.**

Find $\displaystyle\sum_{r=2}^{50}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$, expressing the answer as an exact fraction. **[3]**

This is a method of differences question.

$$\sum_{r=2}^{50}\left(\frac{1}{r-1}-\frac{1}{r+1}\right) = \frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$$

$$+ \dots$$ all sums to zero

$$+ \cancel{\frac{1}{47}} - \cancel{\frac{1}{49}} + \cancel{\frac{1}{48}} - \frac{1}{50} + \cancel{\frac{1}{49}} - \frac{1}{51}$$

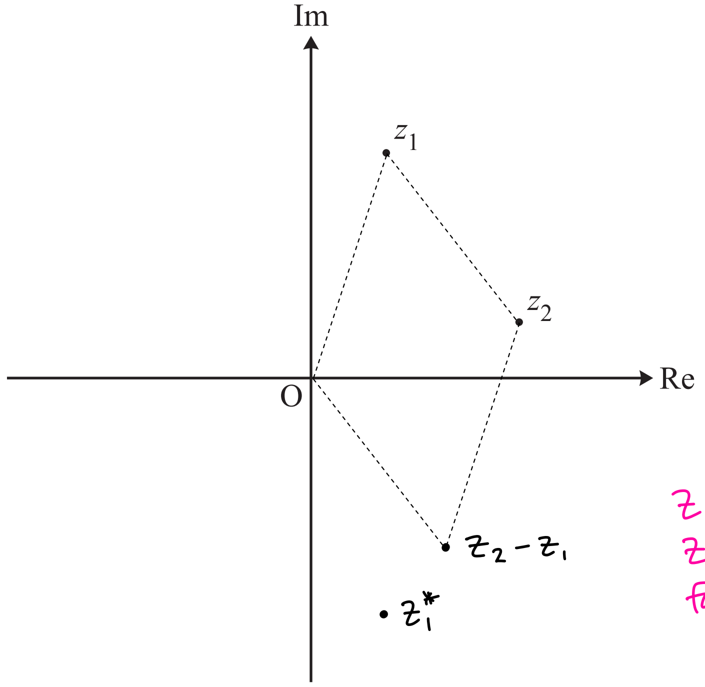
$$= 1 + \frac{1}{2} - \frac{1}{50} - \frac{1}{51}$$

$$= \frac{1862}{1275}$$ give answer in exact form

(a) On the copy of Fig. 2 in the Printed Answer Booklet, mark points representing each of the following complex numbers.

- z_1^*
- $z_2 - z_1$

[2]



$z_2 - z_1$, z_1, z_2 and O form a parallelogram

z^* is the complex conjugate of z_1

G

(b) In this question you must show detailed reasoning.

In the case where $z_1 = 1+2\text{i}$ and $z_2 = 3+\text{i}$, find $\dfrac{z_2 - z_1}{z_1^*}$ in the form $a+\text{i}b$, where a and b are real numbers. **[2]**

First we need to find $z_2 - z_1$ and z_1^*.

$z_2 - z_1 = 3 + i - (1+2i)$

$= 2 - i$

$z_1^* = 1 - 2i$

Now we find the fraction.

so $\dfrac{z_2 - z_1}{z_1^*} = \dfrac{2-i}{1-2i} \times \dfrac{1+2i}{1+2i} = \dfrac{4+3i}{5}$

'realise' the denominator $= \frac{4}{5} + \frac{3}{5}i$

A

3 In this question you must show detailed reasoning.

The roots of the equation $x^2 - 2x + 4 = 0$ are α and β.

(a) Find α and β in modulus-argument form. **[4]**

Here we use the quadratic formula.

using $\sqrt{-1} = i$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4\times1\times4}}{2\times1} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{3}i}{2}$$

$$= 1 \pm \sqrt{3}i$$

Now we find the modulus and argument of α and β.

$|\alpha| = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$ $\left(|z| = \sqrt{a^2+b^2}\right)$

$\arg(1+\sqrt{3}) = \arctan(\sqrt{3}) = \frac{\pi}{3}$ $\left(\arg z = \arctan\frac{b}{a}\right)$

$\arg(1-\sqrt{3}) = \arctan(-\sqrt{3}) - \frac{\pi}{3}$

And write in mod-arg form $(r(\cos\theta + i\sin\theta))$

$2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

A

(b) Hence or otherwise show that α and β are both roots of $x^3 + \lambda = 0$, where λ is a real constant to be determined. **[3]**

Here we need to cube both α and β to find λ.

$$(1+\sqrt{3}i)^3 = 1^3 + 3(1)^2(\sqrt{3}i) + 3(1)(\sqrt{3}i)^2 + (\sqrt{3}i)^3$$
$$= 1 + 3\sqrt{3}i - 9 - 3\sqrt{3}i$$
$$= -8$$
$$(1-\sqrt{3}i)^3 = 1^3 + 3(1)^2(-\sqrt{3}i) + 3(1)(-\sqrt{3}i)^2 + (-\sqrt{3}i)^3$$
$$= 1 - 3\sqrt{3}i - 9 + 3\sqrt{3}i$$
$$= -8$$

So since $\alpha^3 = \beta^3 = -8$, $\lambda = 8$

G

4 The matrix **M** is $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) (i) Calculate det **M**. **[1]**

Use a calculator to find det M.

det M = 1

G

(ii) State two geometrical consequences of this value for the transformation associated with **M**. **[2]**

Since det M > 0, orientation is preserved.

Since det M = 1, the volume scale factor is 1 so volume is preserved.

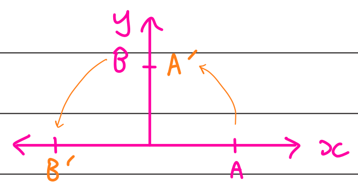
G

(b) Describe fully the transformation associated with **M**. **[1]**

The point on the z-axis does not change position.

Looking down the z-axis,

So M represents a rotation of 90° ACW about the z-axis.



A

5 You are given that $u_1 = 5$ and $u_{n+1} = u_n + 2n + 4$.

Prove by induction that $u_n = n^2 + 3n + 1$ for all positive integers n. **[6]**

Step one : base case

When $n=1$, $U_1 = 5$

$U_1 = (1)^2 + 3(1) + 1 = 5$ $\therefore$ true for $n=1$

Step two : assumption

Assume true for $n=k$, so $U_k = k^2 + 3k + 1$.

Step three : inductive step

Using the assumed result for $n=k$,

$$U_{k+1} = U_k + 2k + 4$$

$$U_{k+1} = k^2 + 3k + 1 + 2k + 4$$

$$U_{k+1} = k^2 + 5k + 5$$

$$= k^2 + 2k + 1 + 3k + 3 + 1$$

$$= (k+1)^2 + 3(k+1) + 1$$

$\therefore$ true for $n = k+1$.

Step four : conclusion

If the result is true for $n=k$, it is true for $n=k+1$. Since it is true for $n=1$, it is true for all positive integer values of n.

A

6 The matrices **M** and **N** are $\begin{pmatrix}\lambda & 2\\ 2 & \lambda\end{pmatrix}$ and $\begin{pmatrix}\mu & 1\\ 1 & \mu\end{pmatrix}$ respectively, where λ and μ are constants.

(a) Investigate whether **M** and **N** are commutative under multiplication. **[2]**

If MN = NM, M and N are commutative under multiplication.

$$MN = \begin{pmatrix}\lambda & 2\\ 2 & \lambda\end{pmatrix}\begin{pmatrix}\mu & 1\\ 1 & \mu\end{pmatrix} = \begin{pmatrix}\lambda\mu+2 & \lambda+2\mu\\ \lambda+2\mu & \lambda\mu+2\end{pmatrix}$$

$$NM = \begin{pmatrix}\mu & 1\\ 1 & \mu\end{pmatrix}\begin{pmatrix}\lambda & 2\\ 2 & \lambda\end{pmatrix} = \begin{pmatrix}\lambda\mu+2 & \lambda+2\mu\\ \lambda+2\mu & \lambda\mu+2\end{pmatrix}$$

$\therefore$ M and N are commutative

G

(b) You are now given that **MN** = **I**.

(i) Write down a relationship between det **M** and det **N**. **[1]**

det I = 1

Since det AB = det A × det B,

det M × det N = 1

A

(ii) Given that $\lambda > 0$, find the exact values of λ and μ. **[3]**

So $\begin{pmatrix}\lambda\mu+2 & \lambda+2\mu\\ \lambda+2\mu & \lambda\mu+2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ $\lambda + 2\mu = 0$

$\Rightarrow \lambda = -2\mu$

each corresponding element must be equal.

$\lambda\mu + 2 = 1 \Rightarrow \lambda\mu = -1$

$\mu \times -2\mu = -1$

$\mu^2 = \frac{1}{2} \Rightarrow \mu = \pm\frac{\sqrt{2}}{2}$

$\lambda > 0$ if $\mu = -\frac{\sqrt{2}}{2}$. So $\lambda = -2 \times -\frac{\sqrt{2}}{2} = \sqrt{2}$

hence $\mu = -\frac{\sqrt{2}}{2}$ and $\lambda = \sqrt{2}$

G

(iii) Hence verify your answer to part **(i)**. **[2]**

Find det M and det N and check the relationship.

det M $= (\sqrt{2})^2 - 4 = -2$

det N $= \left(-\frac{\sqrt{2}}{2}\right)^2 - 1 = -\frac{1}{2}$

det M × det N $= -2 \times -\frac{1}{2} = 1$ as required

R

7 In the quartic equation $2x^4 - 20x^3 + ax^2 + bx + 250 = 0$, the coefficients a and b are real. One root of the equation is $2 + \mathrm{i}$.

Find the other roots. **[7]**

Complex roots come in complex conjugate pairs.

So another root is $2 - i$.

Using $\Sigma\alpha = -\frac{b}{a}$ (sum of roots)

$$2 + i + 2 - i + \alpha + \beta = -\frac{(-20)}{2}$$

$$4 + \alpha + \beta = 10$$

$$\alpha + \beta = 6 \Rightarrow \beta = 6 - \alpha \quad \text{(A)}$$

Using $\Sigma\alpha\beta\gamma\delta = -\frac{d}{a}$ (product of roots)

$$(2+i)(2-i)\alpha\beta = -\frac{250}{2}$$

$$5\alpha\beta = 125$$

$$\alpha\beta = 25 \quad \text{(B)}$$

subbing (A) into (B), $\alpha(6-\alpha) = 25$

$$6\alpha - \alpha^2 = 25$$

$$\alpha^2 - 6\alpha + 25 = 0$$

$$\alpha = \frac{6 \pm \sqrt{(-6)^2 - 4\times1\times25}}{2\times1}$$ using the quadratic formula

$$= \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2}$$

$$= 3 \pm 4i$$

Now just check $3 \pm 4i$ are the last two roots.

If $\alpha = 3 + 4i$, $\beta = 6 - (3 + 4i) = 3 - 4i$

If $\alpha = 3 - 4i$, $\beta = 6 - (3 - 4i) = 3 + 4i$

hence roots are $2 + i$, $2 - i$, $3 + 4i$ and $3 - 4i$.

G

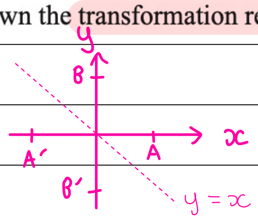
8 **(a)** The matrix $\mathbf{M}$ is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

(i) Find $\mathbf{M}^2$. **[1]**

$$M^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

G

(ii) Write down the transformation represented by $\mathbf{M}$. **[1]**



Reflection in line $y = -x$.

G

(iii) Hence state the geometrical significance of the result of part **(i)**. **[1]**

$M^2 = I$, so

Two reflections in $y = -x$ are equivalent to the identity transformation.

A

(b) The matrix $\mathbf{N}$ is $\begin{pmatrix} k+1 & 0 \\ k & k+2 \end{pmatrix}$, where k is a constant.

Using determinants, investigate whether $\mathbf{N}$ can represent a reflection. **[4]**

First we need to find det N

$$\det N = (k+1)(k+2) - 0$$
$$= k^2 + 3k + 2$$

N represents a reflection when $\det N = -1$ as seen in a). So

$$k^2 + 3k + 2 = -1$$
$$k^2 + 3k + 3 = 0$$

This must have real roots such that $\det N = -1$.

$$\text{determinant} = (3)^2 - 4 \times 1 \times 3$$
$$= 9 - 12 = -3$$

Hence as the determinant < 0, N can never represent a reflection

9 Three planes have equations

$$kx + y - 2z = 0$$
$$2x + 3y - 6z = -5$$
$$3x - 2y + 5z = 1$$

where k is a constant.

Investigate the arrangement of the planes for each of the following cases. If in either case the planes meet at a unique point, find the coordinates of that point.

A

(a) $k = -1$ **[3]**

In tabular form, using calculator

$$\underbrace{\begin{pmatrix} -1 & 1 & -2 \\ 2 & 3 & -6 \\ 3 & -2 & 5 \end{pmatrix}}_{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$$

$\det M = -1 \neq 0$, so always meet at a point.

hence $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -1 & 0 \\ -28 & 1 & -10 \\ -13 & 1 & -5 \end{pmatrix}\begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

so meet at $(-1, 3, 2)$ using calculator

A

(b) $k = \frac{2}{3}$ **[4]**

Again in tabular form,

$$\underbrace{\begin{pmatrix} 2/3 & 1 & -2 \\ 2 & 3 & -6 \\ 3 & -2 & 5 \end{pmatrix}}_{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$$

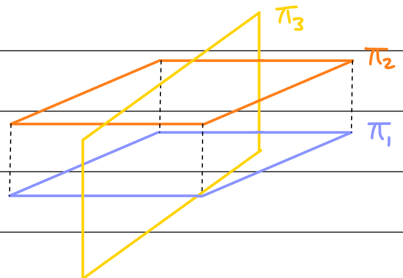
$\det M = 0$, so planes do not meet at a unique point.

let $\Pi_1 : \frac{2}{3}x + y - 2z = 0$

$\Pi_2 : 2x + 3y - 6z = -5$

$\Pi_3 : 3x - 2y + 5z = 1$

notice that $\Pi_2 = 3\Pi_2$. Hence Π_1 and Π_2 are parallel. So Π_1 and Π_2 are parallel planes intersected by Π_3.



R

10 A vector **v** has magnitude 1 unit. The angle between **v** and the positive z-axis is 60°, and **v** is parallel to the plane $x-2y=0$.

Given that $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, where a, b and c are all positive, find **v**. **[7]**

First we can use the magnitude of $\underline{v}$.

Since $|\underline{v}|=1$, $\sqrt{a^2+b^2+c^2}=1$

$\Rightarrow a^2+b^2+c^2=1$ (A)

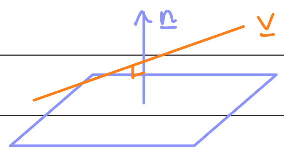
Now we can look at the angle $\underline{v}$ makes with the z-axis. Recall that $\cos\theta=\dfrac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$

Also the z-axis is represented by $\underline{k}$. So

$$\cos 60° = \frac{a(0)+b(0)+c(1)}{\sqrt{a^2+b^2+c^2}\sqrt{1}}$$

$\sqrt{a^2+b^2+c^2}=1$

$\frac{1}{2}=\frac{c}{1} \Rightarrow c=\frac{1}{2}$



If $\underline{v}$ is parallel to the plane $x-2y=0$, $\underline{v}$ is perpendicular to the normal $\underline{i}-2\underline{j}$.

So $\underline{v}\cdot(\underline{i}-2\underline{j})=0$.

$a(1)+b(-2)+c(0)=0$

$a=2b$ (B)

Substituting (B) and $c=\frac{1}{2}$ into (A),

$(2b)^2+b^2+\left(\frac{1}{2}\right)^2=1$

$5b^2=\frac{3}{4}$

$b^2=\frac{3}{20}$

$b=\sqrt{\frac{3}{20}}=0.3872\ldots=0.387$ (3sf)

so $a=2(0.3872\ldots)=0.7745\ldots$

$=0.775$ (3sf)

hence $\underline{v}=0.775\underline{i}+0.387\underline{j}+0.5\underline{k}$ (answer space continued on next page)

10	**(continued)**

ADDITIONAL ANSWER SPACE

If additional space is required, you should use the following lined page(s). The question number(s) must be clearly shown in the margin(s).